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Gröbner Deformations of Hypergeometric Differential Equations

With 14 Figures

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Preface

In recent years, numerous new algorithms for dealing with rings of differential operators have been discovered and implemented. A main tool is the theory of Gröbner bases, which is reexamined in this book from the point of view of geometric deformations. Perturbation techniques have a long tradition in analysis; Gröbner deformations of left ideals in the Weyl algebra are the algebraic analogue to classical perturbation techniques.

The algorithmic methods introduced in this book are aimed at studying the systems of multidimensional hypergeometric partial differential equations introduced by Gel'fand, Kapranov and Zelevinsky. The Gröbner deformation of these GKZ hypergeometric systems reduces problems concerning hypergeometric functions to questions about commutative monomial ideals, and thus leads to an unexpected interplay between analysis and combinatorics.

This book contains original research results on holonomic systems and hypergeometric functions, and it raises many open problems for future research in this rapidly growing area of computational mathematics. An effort has been made to give a presentation which is both accessible to beginning graduate students and attractive to researchers in a variety of mathematical disciplines. The intended audience consists of anyone who is interested in algorithmic mathematics or in mathematical algorithms.

This book project started when the three of us met in Sapporo in August 1997. We had gotten together to work on a joint research paper on topics now contained in Chapter 4. We suddenly realized that we needed more background, almost none of which we could find in the existing literature on D-modules and linear partial differential equations. We then started to develop all the necessary basic material from scratch, and our manuscript soon turned from a draft for a research paper into a draft for a book.

We are grateful to two institutes whose support has been crucial: the Research Institute for Mathematical Sciences (RIMS) at Kyoto University hosted Bernd Sturmfels during the academic year 1997/98, and the Mathematical Sciences Research Institute (MSRI) at Berkeley hosted Nobuki Takayama during the academic year 1998/99. Mutsumi Saito visited his coauthors several times for short periods at both institutes. Bernd Sturmfels also acknowledges partial support from the U.S. National Science Foundation.

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This book is dedicated to our respective families, whose encouragement and support for this enterprise has been invaluable.

June 1999 *Mutsumi Saito, Bernd Sturmfels, Nobuki Takayama*

Contents

1. Basic Notions

This book provides symbolic algorithms for constructing holomorphic solutions to systems of linear partial differential equations with polynomial coefficients. Such a system is represented by a left ideal I in the Weyl algebra

$$D \quad = \quad \mathbf{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle.$$

By a *Gröbner deformation* of the left ideal I we mean an initial ideal $\mathrm{in}_{(-w,w)}(I) \subset D$ with respect to some generic weight vector $w = (w_1, \ldots, w_n)$ with real coordinates w_i. Here the variable x_i has the weight $-w_i$, and the operator ∂_i has the weight w_i, so as to respect the *product rule of calculus*:

$$\partial_i \cdot x_i \quad = \quad x_i \cdot \partial_i + 1.$$

Using techniques from computational commutative algebra, one can determine an explicit solution basis for the Gröbner deformation $\mathrm{in}_{(-w,w)}(I)$. The issue is to extend it to a solution basis of I. This problem is solved in Chapter 2 under the natural hypothesis that the given D-ideal I is *regular holonomic.* This hypothesis is valid for the D-ideals representing *hypergeometric integrals*, whose asymptotic expansions are constructed algorithmically in Chapter 5.

Our main interest lies in the systems of hypergeometric differential equations introduced by Gel'fand, Kapranov and Zelevinsky in the 1980's. Here is a simple, but important, example of a hypergeometric system for $n = 3$:

$$I \quad = \quad D \cdot \{\, \partial_1\partial_3 - \partial_2^2,\ x_1\partial_1 + x_2\partial_2 + x_3\partial_3,\ x_2\partial_2 + 2x_3\partial_3 - 1 \,\}.$$

If $w = (1,0,0)$ then the Gröbner deformation of these equations equals

$$\mathrm{in}_{(-w,w)}(I) \quad = \quad D \cdot \{\, \partial_1\partial_3\, ,\ x_1\partial_1 + x_2\partial_2 + x_3\partial_3,\ x_2\partial_2 + 2x_3\partial_3 - 1 \,\}.$$

It is quite easy to see that the space of solutions to $\mathrm{in}_{(-w,w)}(I)$ is spanned by x_2/x_1 and x_3/x_2. Starting from these two Laurent monomials as w-lowest terms, our algorithm to be presented in Section 2.6 constructs two linearly independent Laurent series solutions to the original system I, namely,

$$-\frac{x_2}{2x_1} \pm \left(\frac{x_2}{2x_1} - \sum_{m=0}^{\infty} \frac{1}{m+1} \binom{2m}{m} \frac{x_1^m x_3^{m+1}}{x_2^{2m+1}} \right) \quad = \quad \frac{-x_2 \pm \sqrt{x_2^2 - 4x_1x_3}}{2x_1}.$$

This is the familiar *quadratic formula* for expressing the two zeros of a quadratic polynomial $p(z) = x_1 z^2 + x_2 z + x_3$ in terms of its three coefficients. It is an amusing challenge to write down the analogous hypergeometric differential equations which annihilate the five roots of the general quintic

$$q(z) \quad = \quad x_1 z^5 + x_2 z^4 + x_3 z^3 + x_4 z^2 + x_5 z + x_6.$$

In this chapter we introduce the topics covered in this book. After treating Gröbner basics in the Weyl algebra, we review the classical Gauss hypergeometric function and how it is expressed in the Gel'fand-Kapranov-Zelevinsky (GKZ) scheme. Section 1.4 gives an introduction to holonomic systems of differential equations from the Gröbner basis point of view, and in Section 1.5 we study a special family of GKZ hypergeometric functions, namely, those which arise by integrating products of linear forms with generic coefficients.

1.1 Gröbner Bases in the Weyl Algebra

Let $\mathbf{k}$ be a field of characteristic zero, typically a subfield of the complex numbers $\mathbf{C}$. The *Weyl algebra* of dimension n is the free associative $\mathbf{k}$-algebra

$$D_n \quad = \quad \mathbf{k}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$

modulo the commutation rules

$$x_i x_j = x_j x_i, \ \ \partial_i \partial_j = \partial_j \partial_i, \ \ \partial_i x_j = x_j \partial_i \ \text{for}\ i \neq j, \ \text{ and } \ \partial_i x_i = x_i \partial_i + 1.$$

If no confusion arises we simply drop the dimension index and write D for D_n. The Weyl algebra is isomorphic to the ring of differential operators on affine n-space $\mathbf{k}^n$. This is proved, for instance, in Coutinho's excellent text book on the Weyl algebra [26, Theorem 2.3, p.23]. The natural action of the Weyl algebra D on polynomials $f \in \mathbf{k}[x_1, \ldots, x_n]$ is as follows:

$$\partial_i \bullet f = \frac{\partial f}{\partial x_i}\ , \qquad x_i \bullet f = x_i f. \tag{1.1}$$

Since $\mathbf{k}[x_1, \ldots, x_n]$ is also a subring of Weyl algebra D, the symbol $\bullet$ helps distinguish the action (1.1) from the product $\cdot : D \times D \to D$. For instance,

$$\partial_1^2 \bullet x_1^4 \ = \ 12 x_1^2 \quad \text{but} \quad \partial_1^2 \cdot x_1^4 \ = \ x_1^4 \partial_1^2 + 8 x_1^3 \partial_1 + 12 x_1^2.$$

The Weyl algebra D acts by the same rule (1.1) on many $\mathbf{k}[x_1, \ldots, x_n]$-modules F, including *formal power series* $F = \mathbf{k}[[x_1, \ldots, x_n]]$, or, if $\mathbf{k} \subseteq \mathbf{C}$, *holomorphic functions* $F = \mathcal{O}^{\mathrm{an}}(U)$ on an open subset U of $\mathbf{C}^n$.

A system of linear differential equations with polynomial coefficients can be identified with a left ideal in D. Suppose that we are given a system of linear differential equations for an unknown function $u = u(x_1, \ldots, x_n)$,

$$L_1 \bullet u = 0, \ \ldots, L_m \bullet u = 0, \ L_i \in D.$$

Then, the unknown function u also satisfies the differential equation

$$\sum_{i=1}^{m} (c_i L_i) \bullet u = 0$$

for any elements c_i in D. This implies that the system of differential equations may be expressed as

$$L \bullet u = 0, \quad L \in I$$

where I is the left ideal in D generated by $L_1, \ldots, L_m$. This point of view enables us to study differential equations through Gröbner bases for left ideals in the Weyl algebra.

Any element p of D has a unique *normally ordered expression*

$$p \;=\; \sum_{(\alpha,\beta)\in E} c_{\alpha\beta} \cdot x^\alpha \partial^\beta, \tag{1.2}$$

where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$, $c_{\alpha\beta} \in \mathbf{k}^* = \mathbf{k}\backslash\{0\}$, and E is a finite subset of $\mathbf{N}^{2n}$. Here, $\mathbf{N} = \{0, 1, 2, \ldots\}$. In other words, we have the following natural $\mathbf{k}$-vector space isomorphism between the commutative polynomial ring in $2n$ variables and the Weyl algebra:

$$\Psi \,:\, \mathbf{k}[x,\xi] = \mathbf{k}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n] \;\to\; D\,,\; x^\alpha \xi^\beta \mapsto x^\alpha \partial^\beta. \tag{1.3}$$

When doing calculations in the Weyl algebra – by hand or by computer – the isomorphism Ψ provides a useful representation of the elements. Efficient multiplication in D can be accomplished by the following *Leibnitz formula*:

Theorem 1.1.1. *For any two polynomials f and g in $\mathbf{k}[x,\xi]$ we have*

$$\Psi(f) \cdot \Psi(g) \quad = \quad \sum_{k_1,\ldots,k_n \geq 0} \frac{1}{k_1! \cdots k_n!} \cdot \Psi\left(\frac{\partial^k f}{\partial \xi^k} \cdot \frac{\partial^k g}{\partial x^k}\right).$$

Proof. Both the left hand side and the right hand side are $\mathbf{k}$-bilinear, so we may assume that f and g are monomials, say, $f = x^\alpha \xi^\beta$ and $g = x^\gamma \xi^\delta$. Clearly, we can factor out x^α and ξ^δ on both sides, so we may assume $f = \xi^\beta$ and $g = x^\gamma$. Both sides of the desired equation can be written as a product,

$$\prod_{i=1}^{n} (\Psi(\xi_i^{\beta_i})\Psi(x_i^{\gamma_i})) \quad = \quad \prod_{i=1}^{n} \sum_{k_i \geq 0} \frac{1}{k_i!} \cdot \Psi\left(\frac{\partial^{k_i} \xi_i^{\beta_i}}{\partial \xi_i^{k_i}} \cdot \frac{\partial^{k_i} x_i^{\gamma_i}}{\partial x_i^{k_i}}\right).$$

Hence it suffices to prove the case $n = 1$, which amounts to the formula

$$\partial^i x^j \;=\; \sum_{k=0}^{\min\{i,j\}} \frac{i(i-1)\cdots(i-k+1)\, j(j-1)\cdots(j-k+1)}{k\,!}\, x^{j-k} \partial^{i-k}. \tag{1.4}$$

This formula can be derived from $\partial x = x\partial + 1$ by induction on i and j. □

One remark on formula (1.4): throughout this book we freely use the convention that x abbreviates x_1 and ∂ abbreviates ∂_1 in the case $n = 1$.

A real vector $(u, v) = (u_1, \ldots, u_n, v_1, \ldots, v_n) \in \mathbf{R}^{2n}$ is called a *weight vector* (for the Weyl algebra) if

$$u_i + v_i \geq 0 \qquad \text{for} \quad i = 1, 2, \ldots, n.$$

Here u_i is the weight of the generator x_i, and v_i is the weight of the generator ∂_i. This condition will always be assumed in this book. The *associated graded ring* $\mathrm{gr}_{(u,v)}(D)$ of the Weyl algebra D with respect to a weight vector (u, v) is the $\mathbf{k}$-algebra generated by

$$\{x_1, \ldots, x_n\} \cup \{\partial_i \,:\, u_i + v_i = 0\} \cup \{\xi_i \,:\, u_i + v_i > 0\}$$

with all variables commuting with each other except for $\partial_i x_i = x_i \partial_i + 1$. In fact, when u_i, v_i are integers, $\mathrm{gr}_{(u,v)}(D)$ is the associated graded ring of D with respect to the filtration $\cdots \subset F_0 \subset F_1 \subset \cdots$ defined by

$$F_m \quad = \quad \left\{ \sum_{u\alpha + v\beta \leq m} c_{\alpha\beta} x^\alpha \partial^\beta \right\}.$$

The two extreme cases of this definition are

$$\begin{aligned} \mathrm{gr}_{(u,v)}(D) &= \mathbf{k}[x, \xi] && \text{if each coordinate of } u + v \text{ is positive;} \\ \mathrm{gr}_{(u,v)}(D) &= \;D && \text{if } u + v \text{ is the zero vector.} \end{aligned}$$

For a non-zero element p in the Weyl algebra D we define the *initial form* $\mathrm{in}_{(u,v)}(p)$ of p with respect to (u, v) as follows. Let $m = \max_{(\alpha,\beta)\in E}(\alpha{\cdot}u + \beta{\cdot}v)$, select the terms of maximum weight m in the normally ordered expression (1.2), and then replace ∂_i by ξ_i for all i with $u_i + v_i > 0$. In symbols,

$$\mathrm{in}_{(u,v)}(p) \quad = \sum_{\substack{(\alpha,\beta)\in E \\ \alpha\cdot u + \beta\cdot v = m}} c_{\alpha\beta} \prod_{i:u_i+v_i>0} x_i^{\alpha_i} \xi_i^{\beta_i} \prod_{i:u_i+v_i=0} x_i^{\alpha_i} \partial_i^{\beta_i} \quad \in \quad \mathrm{gr}_{(u,v)}(D).$$

For $p = 0$, we define $\mathrm{in}_{(u,v)}(p) = 0$.

A left ideal I in the Weyl algebra D will be called a *D-ideal.* The following result is an important consequence of the Leibnitz formula (Theorem 1.1.1).

Corollary 1.1.2. *Let I be a D-ideal and (u, v) any weight vector. Then the $\mathbf{k}$-vector space*

$$\mathrm{in}_{(u,v)}(I) \quad := \quad \mathbf{k} \cdot \{\, \mathrm{in}_{(u,v)}(\ell) \mid \ell \in I \,\}$$

is a left ideal in the associated graded ring $\mathrm{gr}_{(u,v)}(D)$.

Definition 1.1.3. The ideal $\mathrm{in}_{(u,v)}(I)$ in $\mathrm{gr}_{(u,v)}(D)$ is called the *initial ideal* of a D-ideal I with respect to the weight vector (u,v). A finite subset G of D is a *Gröbner basis* of I with respect to (u,v) if I is generated by G and $\mathrm{in}_{(u,v)}(I)$ is generated by initial forms $\mathrm{in}_{(u,v)}(g)$ where g runs over G, i.e.,

$$I = D \cdot G \quad \text{and} \quad \mathrm{in}_{(u,v)}(I) = \mathrm{gr}_{(u,v)}(D) \cdot \mathrm{in}_{(u,v)}(G), \tag{1.5}$$

$\mathrm{in}_{(u,v)}(G) := \{\, \mathrm{in}_{(u,v)}(g) \mid g \in G\}$.

Note that if $u + v > 0$ then $\mathrm{in}_{(u,v)}(I)$ is an ideal in the commutative polynomial ring $\mathbf{k}[x,\xi]$, while the Gröbner basis G is still a subset of the Weyl algebra D. The following examples will clarify the above definitions.

Example 1.1.4. Let $n = 1$, $(u,v) = (-1,2)$, $F_1 = \{x^3\partial^2, x\partial^4\}$, and $I = DF_1$. The singleton $G = \{\partial^2\}$ is a Gröbner basis for I, and the initial ideal equals $\mathrm{in}_{(u,v)}(I) = \langle \xi^2 \rangle$. (In this book we use "$\langle \ldots \rangle$" for commutative polynomial ideals.) The singleton $F_2 = \{\partial^2 - x\partial^2\}$ satisfies the second condition of (1.5) but fails the first, while F_1 satisfies the first condition but fails the second. Indeed, a collection of normally ordered monomials is rarely a Gröbner basis.

It is our next goal to describe the Buchberger algorithm for computing Gröbner bases in the Weyl algebra. To this end we need to specify a total order $\prec$ on the set of normally ordered monomials $x^\alpha\partial^\beta$ in D. Such an order is called a *multiplicative monomial order* if the following two conditions hold:

1. $1 \prec x_i\partial_i$ for $i = 1, 2, \ldots, n$;
2. $x^\alpha\partial^\beta \prec x^a\partial^b$ implies $x^{\alpha+s}\partial^{\beta+t} \prec x^{a+s}\partial^{b+t}$ for all $(s,t) \in \mathbf{N}^{2n}$.

A multiplicative monomial order $\prec$ is called a *term order* (for the Weyl algebra) if $1 = x^0\partial^0$ is the smallest element of $\prec$. A multiplicative monomial order which is not a term order (sometimes called a *non-term order*) has infinite strictly decreasing chains but a term order does not (see, e.g., [27, p.70, Cor.6]). For information on frequently used term orders (*lexicographic order*, *reverse lexicographic order*, *elimination order*, *graded reverse lexicographic order*) see any book on Gröbner bases, e.g., [1], [12], [27].

The first condition $1 \prec x_i\partial_i$ in the above definition is a consequence of the relation $\partial_i x_i = x_i\partial_i + 1$. Without this assumption the order will not be compatible with multiplication; i.e. we do not have $\mathrm{in}_\prec(fg) = \mathrm{in}_\prec(f) \cdot \mathrm{in}_\prec(g)$.

Example 1.1.5. Let $n = 1$. Let $\prec$ be the total order defined by $x^\alpha\partial^\beta \prec x^a\partial^b \Leftrightarrow \beta - \alpha < b - a$ or ($\beta - \alpha = b - a$ and $\alpha > a$). This is not a multiplicative monomial order and it is not compatible with the multiplication. For instance, $x\partial \prec 1$, and the initial term of $\partial \cdot x\partial = x\partial^2 + \underline{\partial}$ with respect to the order $\prec$ is equal to $\partial \cdot 1$. (In this book, the underline __ will be used to mark the initial term with respect to a given order or the initial form for a given weight.)

Fix a multiplicative monomial order $\prec$. The *initial monomial* $\mathrm{in}_\prec(p)$ of an element $p \in D$ is the commutative monomial $x^\alpha \xi^\beta$ in $\mathbf{k}[x,\xi]$ such that $x^\alpha \partial^\beta$ is the $\prec$-largest normally ordered monomial in the expansion (1.2) of p. For a finite set F in D, we define $\mathrm{in}_\prec(F) = \{\, \mathrm{in}_\prec(f) \,|\, f \in F \,\}$. For a D-ideal I we define the *initial ideal* $\mathrm{in}_\prec(I)$ to be the monomial ideal in $\mathbf{k}[x,\xi]$ generated by $\{\mathrm{in}_\prec(p) \,|\, p \in I\}$. A finite subset G of D is said to be a *Gröbner basis* of I with respect to $\prec$ if I is generated by G and $\mathrm{in}_\prec(I)$ is generated by the (commutative) monomials $\mathrm{in}_\prec(g)$ where g runs over G.

The definitions in the previous paragraph extend naturally to elements and ideals in the associated graded rings $\mathrm{gr}_{(u,v)}(D)$. In particular, we shall make frequent use of ordinary commutative Gröbner bases in $\mathbf{k}[x,\xi]$ with respect to term orders $\prec$. This will be relevant for part 2 in the next theorem.

As it now stands, there are two notions of Gröbner bases in D, one for weight vectors and one for multiplicative monomial orders. Theorem 1.1.6 will relate these two. Let $(u,v) \in \mathbf{R}^{2n}$ be a weight vector, and let $\prec$ be any term order. Then we define a multiplicative monomial order $\prec_{(u,v)}$ as follows:

$$x^\alpha \partial^\beta \prec_{(u,v)} x^a \partial^b \Leftrightarrow \alpha u + \beta v < au + bv \text{ or}$$
$$(\alpha u + \beta v = au + bv \text{ and } x^\alpha \partial^\beta \prec x^a \partial^b).$$

Note that $\prec_{(u,v)}$ is a term order if and only if (u,v) is a non-negative vector.

Theorem 1.1.6. *Let I be a D-ideal, $(u,v) \in \mathbf{R}^{2n}$ any weight vector, $\prec$ any term order, and G a Gröbner basis for I with respect to $\prec_{(u,v)}$. Then*

(1) *the set G is a Gröbner basis for I with respect to (u,v), and*
(2) *the set $\mathrm{in}_{(u,v)}(G)$ is a Gröbner basis for $\mathrm{in}_{(u,v)}(I)$ with respect to $\prec$.*

Proof. Suppose that G is not a Gröbner basis for I with respect to (u,v). Then there exists an element $f \in I$ whose initial form $\mathrm{in}_{(u,v)}(f)$ is not in the left ideal generated by $\mathrm{in}_{(u,v)}(G)$. Since $\prec$ is a term order, and thus has no infinite descending chains, we may further assume that the initial monomial

$$\mathrm{in}_\prec\big(\mathrm{in}_{(u,v)}(f)\big) \quad = \quad \mathrm{in}_{\prec_{(u,v)}}(f) \quad \in \quad \mathbf{k}[x,\xi] \tag{1.6}$$

is minimal with respect to $\prec$ among all elements f with this property. By our assumption there exists $g \in G$ such that $\mathrm{in}_{\prec_{(u,v)}}(g)$ divides (1.6). We can choose $c \in \mathbf{k}^*$ and $\alpha, \beta \in \mathbf{N}^n$ such that $f' := f - c x^\alpha \partial^\beta \cdot g$ has $\prec_{(u,v)}$-leading monomial $\prec$-smaller than (1.6). This implies that $\mathrm{in}_{(u,v)}(f') = \mathrm{in}_{(u,v)}(f) - c \cdot \mathrm{in}_{(u,v)}(x^\alpha \partial^\beta \cdot g)$ is not in the left ideal generated by $\mathrm{in}_{(u,v)}(G)$. This is a contradiction to the $\prec$-minimality of (1.6). Part 1 is proved.

For the proof of part 2 we consider an arbitrary (u,v)-homogeneous element $h \in \mathrm{in}_{(u,v)}(I)$. By Corollary 1.1.2 there exists $f \in I$ such that $h = \mathrm{in}_{(u,v)}(f)$. Formula (1.6) shows that $\mathrm{in}_\prec(h)$ lies in the monomial ideal generated by $\mathrm{in}_{\prec_{(u,v)}}(G)$. Moreover, since $\prec$ is a term order, we may conclude (for instance, by Theorem 1.1.7 below) that $\mathrm{in}_{(u,v)}(G)$ actually generates $\mathrm{in}_{(u,v)}(I)$ as a left ideal in $\mathrm{gr}_{(u,v)}(D)$. □

Theorem 1.1.6 reduces the problem of computing Gröbner bases with respect to weight vectors (u, v) to the problem of computing Gröbner bases with respect to multiplicative monomial orders. We will divide our discussion of that problem in two steps. First we study the case where $\prec$ is a term order, and next we introduce the homogenized Weyl algebra to solve the case where $\prec$ is a non-term order. The former will be carried out in this section and the latter in the next section. The latter case includes the most interesting orders $\prec_{(-w,w)}$ which arise geometrically from the action of the algebraic torus $(\mathbf{k}^*)^n$ on the Weyl algebra (see Section 2.3).

Theorem 1.1.7. *Let $\prec$ be a term order and G a Gröbner basis for its D-ideal $I = D \cdot G$ with respect to $\prec$. Any element f in I admits a standard representation in terms of G: there exist $c_1, \ldots, c_m \in D$ that satisfy*

$$f = \sum_{j=1}^{m} c_j g_j, \quad \textit{where} \quad g_j \in G \quad \textit{and} \quad \mathrm{in}_\prec(c_j g_j) \preceq \mathrm{in}_\prec(f) \textit{ for all } j.$$

This implies that the first condition in (1.5) can be weakened to $G \subset I$ if we assume $(u, v) > 0$.

The proof of Theorem 1.1.7 is analogous to the familiar commutative case (see, e.g., [1], [12], [27], [32]). What we must do here, however, is to carefully define S-pairs and present the normal form algorithm in the Weyl algebra D. We fix a multiplicative monomial order $\prec$. For two normally ordered elements

$$\begin{aligned} f &= f_{\alpha\beta} x^\alpha \partial^\beta + \text{lower order terms with respect to } \prec, \\ g &= g_{ab} x^a \partial^b + \text{lower order terms with respect to } \prec \end{aligned}$$

in D, we define the *S-pair* of f and g by

$$\mathrm{sp}(f, g) \quad = \quad x^{\alpha'} \partial^{\beta'} f - (f_{\alpha\beta}/g_{ab}) x^{a'} \partial^{b'} g$$

where $\alpha'_i = \max(\alpha_i, a_i) - \alpha_i$, $\beta'_i = \max(\beta_i, b_i) - \beta_i$, $a'_i = \max(\alpha_i, a_i) - a_i$, $b'_i = \max(\beta_i, b_i) - b_i$. The multipliers for f and g are chosen to cancel the initial monomials of f and g. Note that we have used the condition $x_i \partial_i \succ 1$ to cancel the initial terms. We say that $x^\alpha \partial^\beta$ is *divisible* by $x^a \partial^b$ if $\alpha_i \geq a_i$ and $\beta_i \geq b_i$ for all i.

Let us introduce a normal form algorithm in D:

normalForm$_\prec(f, \{g_1, \ldots, g_m\})$:=
    $r := f$
    while (in$_\prec(r)$ is divisible by an in$_\prec(g_i)$) {
        $r := \mathrm{sp}(r, g_i)$ (1.7)
    }
    $r := \mathrm{in}_\prec(r)_{|\xi \to \partial}$
        $+\mathrm{normalForm}_\prec(r - \mathrm{in}_\prec(r)_{|\xi\to\partial}, \{g_1, \ldots, g_m\})$ (1.8)
    return(r)

Here $\text{in}_\prec(r)_{|\xi\to\partial}$ means $\Psi(\text{in}_\prec(r))$, with Ψ as defined in (1.3). Note that we use a recursive call at (1.8). It is used for presenting the algorithm compactly and has no deep meaning. The output f' of this algorithm is a *normal form* of f by $G = \{g_1, \ldots, g_m\}$. The normal form algorithm is also called a *division algorithm.* When $\prec$ is a term order, this normal form algorithm terminates. The normal form is not always unique for general G, but when G is a Gröbner basis with respect to a term order $\prec$, the normal form is unique. In particular, the normal form of $f \in I$ is 0 by any Gröbner basis G of I.

Example 1.1.8. We present two examples of the computation of normal forms to clarify our definitions. Put $G = \{g_1, g_2\}$, $g_1 = (x\partial - 2)(x\partial - 4) = x^2\partial^2 - 5x\partial + 8$, $g_2 = (x\partial - 1)\partial^3 = x\partial^4 - \partial^3$ and let $\prec$ be the lexicographic order so that $x \prec \partial$. Then, for example, we have

$$\text{normalForm}_\prec(x^2\partial^5, \{g_1, g_2\}) = 0$$

and

$$\text{normalForm}_\prec(\partial^3 + x^2\partial^2, \{g_1, g_2\}) = \partial^3 + 5x\partial - 8.$$

In fact, we have

$$\begin{aligned} & x^2\partial^5 \\ \longrightarrow & \boxed{x^2\partial^5} - \partial^3 g_1 = -x\partial^4 + \partial^3 \\ \longrightarrow & (\boxed{-x\partial^4} + \partial^3) + g_2 = 0 \end{aligned}$$

and

$$\begin{aligned} & \partial^3 + x^2\partial^2 \\ \longrightarrow & (\partial^3 + \boxed{x^2\partial^2}) - g_1 = \partial^3 + 5x\partial - 8. \end{aligned}$$

Here, we marked by the boxes $\boxed{\cdots}$ terms that will be reduced. Note that our normal form algorithm reduces lower order terms that are reducible.

Proof (of Theorem 1.1.7). We apply the normal form algorithm to f and G. Suppose $f \neq 0$. Since $f_0 := f \in I$ and G is a Gröbner basis, there exists g_{i_1} in G such that $\text{in}_\prec(f)$ is divisible by $\text{in}_\prec(g_{i_1})$. Put $f_1 := \text{sp}(f, g_{i_1}) = f_0 - m_1 g_{i_1}$. Then, we have $f_1 \in I$, $\text{in}_\prec(f_1) \prec \text{in}_\prec(f_0)$, and $\text{in}_\prec(m_1 g_{i_1}) \preceq \text{in}_\prec(f_0)$. We can repeat this procedure and obtain a sequence $f_j := f_{j-1} - m_j g_{i_j}$. Since $\prec$ is a term order, this procedure terminates; there exists J such that $f_J = 0$. The sum of these $m_j g_{i_j}$ gives a standard representation. □

Under our definitions of term orders, S-pairs, and the normal form algorithm, the Gröbner basis can be obtained in an analogous way to the commutative case. The Buchberger algorithm to obtain a *Gröbner basis* can be described as follows when $\prec$ is a term order.

Algorithm 1.1.9 (Buchberger's Algorithm in the Weyl algebra)

Input: $F = \{f_1, \ldots, f_m\}$: a subset of D, $\prec$: a term order.
Output: G : A Gröbner basis for $D \cdot F$ with respect to $\prec$.
pair := $\{(f_i, f_j) \mid 1 \le i < j \le m\}$
$G := F$
while (pair $\neq \emptyset$) {
  Take any element (f, f') from the set pair. (1.9)
  pair := pair $\setminus \{(f, f')\}$
  $h := \mathrm{sp}(f, f')$
  $r :=$ normalForm$_\prec(h, G)$ (1.10)
  if ($r \neq 0$) {
    pair := pair $\cup \{(g, r) \mid g \in G\}$
    $G := G \cup \{r\}$
  }
}
return(G)

Theorem 1.1.10. *Let* $F = \{f_1, \ldots, f_m\}$ *be a finite subset of* D*. Assume that* $\prec$ *is a term order.*

(1) (S-pair criterion) *The set* F *is a Gröbner basis of* $I = D \cdot F$ *with respect to* $\prec$ *if and only if for all pairs* $i \neq j$*, the normal form of the S-pair* $\mathrm{sp}(f_i, f_j)$ *by* F *is zero.*
(2) *The Buchberger algorithm terminates and outputs a Gröbner basis of* I *with respect to* $\prec$.

The proof is analogous to the commutative case. See, e.g., [1, p.40, Theorem 1.7.4], [12, p.211, Theorem 5.48], [27, p.82, Theorem 6].

Example 1.1.11. For $n = 1$, consider $I = D \cdot \{\partial^2, x\partial - 1\}$. Let $\prec$ be any term order. Then, $G = \{\partial^2, x\partial - 1\}$ is a Gröbner basis, because

$$\mathrm{sp}(\underline{\partial^2}, \underline{x\partial} - 1) = \underline{x\partial^2} - \partial(\underline{x\partial} - 1) = x\partial^2 - (x\partial + 1)\partial + \partial = 0.$$

Figure 1.1 pictures the monomials $x^a\xi^b$ which are divisible by an element of $\mathrm{in}_\prec(G)$. The monomials $x^c\xi^d$ which do not lie in $\mathrm{in}_\prec(I)$, or their Ψ-preimages $x^c\partial^d$, are called the *standard monomials* of $\mathrm{in}_\prec(I)$. In other words, a monomial $x^c\xi^d$ which is not divisible by any element of $\mathrm{in}_\prec(G)$ is a standard monomial. The standard monomials are the lattice points in the first orthant which are not dotted.